On a Linear Differential Equation of the Second Order.

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I.

The differential equation to be studied is

$$\frac{d^2y}{dx^2} + \frac{A_{11}x^2 + A_{12}x + A_{13}}{x(1-x^2)}\frac{dy}{dx} + \frac{A_{21}x^4 + A_{22}x^3 + A_{23}x^2 + A_{24}x + A_{25}}{x^2(1-x^2)^2}y = 0,$$

or for brevity

$$\frac{d^2y}{dx^2} = P_1\frac{dy}{dx} + P_2 y.$$

The coefficients P_1 and P_2 have for points of (polar) discontinuity

$$x = 0, \ x = 1, \ x = -1.$$

The domain of the point $x = 0$ is a circle of radius unity having the origin as center, that of the point $x = 1$ is a circle of radius unity having the point $x = 1$ as center, and similarly for the point $x = -1$. The domain of the point $x = \infty$ is the entire infinite plane lying outside the circle having the origin as center and radius unity. These several domains will be denoted by C_0, C_1 C_{-1}, C_∞, and the portion of the plane common to two domains C_i and C_j will be denoted by C_{ij}. The fundamental integrals of the equation in the domains C_i and C_j must coincide in the common domain C_{ij}.

In the domain of the point $x = 0$ write the differential equation in the form

$$\frac{d^2y}{dx^2} = \frac{Q_1}{x}\frac{dy}{dx} + \frac{Q_2}{x^2}y.$$

The fundamental determinant equation for the point $x = 0$ is

$$r(r-1) - rQ_1(0) - Q_2(0) = 0,$$

or, on substituting for $Q_1(0)$ and $Q_2(0)$ their values

$$r(r-1) + rA_{13} + A_{25}$$

the roots of which are

$$r_1 = \tfrac{1}{2}\{(1-A_{13}) + \sqrt{(1-A_{13})^2 - 4A_{25}}\}$$
$$r_2 = \tfrac{1}{2}\{(1-A_{13}) - \sqrt{(1-A_{13})^2 - 4A_{25}}\}.$$

Assume as a special case $A_{13} = 1$, $A_{25} = 0$, then the roots are each equal to zero, and consequently the fundamental integrals of the equation in the domain of $x = 0$ are
$$y_1 = \phi_{11}(x),$$
$$y_2 = \phi_{21}(x) + \phi_{22}(x) \log x.$$
Where ϕ_{11}, ϕ_{21}, ϕ_{22} are, in the domain C_0, uniform and continuous functions of x and do not vanish for $x = 0$; and where ϕ_{22} differs only by a constant factor from ϕ_{11}.

Take the first integral and write for brevity $\phi_{11} = u$. Being a uniform and continuous function of x in the region of the point $x = 0$ and not vanishing for $x = 0$, we have for u the form
$$u = \sum_{a=0}^{a=\infty} C_a x^a,$$
a convergent series in which C_0 is not equal to zero. Substitute u in the differential equation and develop $(1 - x^2)^{-1}$ and $(1 - x^2)^{-2}$ in series going according to ascending powers of x. The product of $\dfrac{du}{dx}$ and the development of $(1 - x^2)^{-1}$ is
$$\sum_{n=0}^{n=\infty} A_n x^n,$$
where
$$A_n = \sum_{k=1}^{k=n+1} \frac{1 + e^{(n+k+1)i\pi}}{2} k C_k,$$
and multiplying this by the factor
$$-(A_{11} x^2 + A_{12} x + 1),$$
we have for the first term in the right-hand side of the differential equation
$$x \frac{d^2 u}{dx^2} = -\frac{(A_{11} x^2 + A_{12} x + 1)}{1 - x^2} \frac{du}{dx} - \frac{(A_{21} x^3 + A_{22} x^2 + A_{23} x + A_{24})}{(1 - x^2)^2} u$$
the value
$$-\sum_{n=0}^{n=\infty} [A_{11} A_{n-2} + A_{12} A_{n-1} + A_n] x^n.$$
Multiplying u by the development of $(1 - x^2)^{-2}$, we have
$$\sum_{n=0}^{n=\infty} x^n \sum_{k=0}^{k=n} \frac{1 + e^{(n+k)i\pi}}{2} \left(\frac{n-k}{2} + 1\right) C_k,$$
or, writing for brevity
$$\sum_{k=0}^{k=n} \frac{1 + e^{(n+k)i\pi}}{2} \left(\frac{n-k}{2} + 1\right) C_k = B_n,$$

simply
$$\sum_{n=0}^{n=\infty} B_n x^n.$$

On multiplying by the remaining factor
$$-(A_{21}x^3 + A_{22}x^2 + A_{23}x + A_{24}),$$
the second term on the right-hand side of the differential equation is
$$-\sum_{n=0}^{n=\infty} \left[A_{21}B_{n-3} + A_{22}B_{n-2} + A_{23}B_{n-1} + A_{24}B_n\right] x^n.$$

The whole equation is now
$$\sum_{n=0}^{n=\infty} n(n+1)C_{n+1}x^n = -\sum_{n=0}^{n=\infty} \left[A_{11}A_{n-2} + A_{12}A_{n-1} + A_n\right] x^n$$
$$-\sum_{n=0}^{n=\infty} \left[A_{21}B_{n-3} + A_{22}B_{n-2} + A_{23}B_{n-1} + A_{24}B_n\right] x^n.$$

Equating the coefficients of x^n in this and we obtain a series of equations for the determination of $C_0, C_1, C_2 \ldots$. It is clear that ultimately every C will be merely C_0 multiplied by a determinate constant. We have for C_{n+1} after transposing one term to the left-hand side of the equation

$$(n+1)^2 C_{n+1} = -A_{21} \sum_{k=0}^{k=n-3} \frac{1 + e^{(n-3+k)i\pi}}{2} \left(\frac{n-3+k}{2} + 1\right) C_k$$
$$-A_{22} \sum_{0}^{n-2} \frac{1 + e^{(n-2+k)i\pi}}{2} \left(\frac{n-2+k}{2} + 1\right) C_k$$
$$-A_{23} \sum_{0}^{n-1} \frac{1 + e^{(n-1+k)i\pi}}{2} \left(\frac{n-1+k}{2} + 1\right) C_k$$
$$-A_{24} \sum_{0}^{n} \frac{1 + e^{(n+k)i\pi}}{2} \left(\frac{n+k}{2} + 1\right) C_k$$
$$-A_{11} \sum_{1}^{n-1} \frac{1 + e^{(n-1+k)i\pi}}{2} k C_k$$
$$-A_{12} \sum_{1}^{n} \frac{1 + e^{(n+k)i\pi}}{2} k C_k$$
$$- \sum_{1}^{n} \frac{1 + e^{(n+k+1)i\pi}}{2} k C_k.$$

In particular,

$$C_1 = - A_{24} C_0$$
$$C_2 = \tfrac{1}{4} \left[- A_{23} + A_{24}^2 + A_{12} A_{24} \right] C_0$$
$$C_3 = \tfrac{1}{36} \left[2 A_{12} A_{23} - 3 A_{12} A_{24}^2 - 2 A_{12}^2 A_{24} + 5 A_{23} A_{24} - A_{24}^3 - 4 A_{11} A_{24} - 4 A_{24} - 4 A_{22} \right],$$

etc.

If we assume

$$A_{24} = 0, \ A_{23} = - 1, \ A_{22} = 0, \ A_{21} = 1, \ A_{13} = 1, \ A_{12} = 0, \ A_{11} = - 3,$$

the differential equation becomes

$$\frac{d^2 y}{dx^2} + \frac{1 - 3x^2}{x (1 - x^2)} \frac{dy}{dx} - \frac{1}{1 - x^2} y = 0,$$

which has for one integral the elliptic integral, K, of the first kind with modulus x. The values of the constants (computed from the general formula) are readily found to be

$$C_0, \ C_1 = 0, \ C_2 = C_0 \tfrac{1}{4}, \ C_3 = 0, \ C_4 = C_0 \frac{1^2 . 3^2}{2^2 . 4^2}, \ C_5 = 0, \ C_6 = C_0 \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2}, \ \text{etc.}$$

Assuming then $C_0 = \dfrac{\pi}{2}$ we have for y_1 the value

$$y_1 = \frac{\pi}{2} \left[1 + \frac{1^2}{2^2} . x^2 + \frac{1^2 . 3^2}{2^2 . 4^2} x^4 + \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} x^6 + \cdots \right]$$

which is the known development of K in terms of the modulus.

I have similarly verified the general formula in the case of the differential equation, giving the elliptic integral of the second kind.

Another relation between five consecutive coefficients is at once obtained if we multiply the given equation through by $x (1 - x^2)^2$ and then equate coefficients of x^n: this is

$$(n + 1)^2 C_{n+1} + (n A_{12} + A_{24}) C_n + \left[- 2 (n - 2)(n - 1) + (n - 1)(A_{11} - 1) + A_{23} \right] C_{n-1}$$
$$+ \left[A_{22} - (n - 2) A_{12} \right] C_{n-2} + \left[(n - 3)(n - 4) - (n - 3) A_{11} + A_{21} \right] C_{n-3} = 0,$$

or say $\Phi (n, C) = 0$. Considered as a function of n we may write

$$\frac{d\Phi (n, C)}{dn} = 2 (n + 1) C_{n+1} + A_{12} C_n + \left[1 + A_{11} - 4 (n - 1) \right] C_{n-1}$$
$$- A_{12} C_{n-2} + \left[2 (n - 3) - (1 + A_{11}) \right] C_{n-3}.$$

For the second integral of the equation we have

$$y_2 = \phi (x) + u \log x,$$

where $\phi (x)$ is a uniform continuous function of x in the domain C_0 and does not vanish for $x = 0$, we have then

$$\phi (x) = \sum_{n=0}^{n=\infty} c_n x^n.$$

Substituting in the differential equation and equating coefficients of x^n, we find as before the relation

$$2(n+1)\, C_{n+1} + A_{12} C_n + [1 + A_{11} - 4\,(n-1)]\, C_{n-1} - A_{12} C_{n-2}$$
$$+ \,[2\,(n-3) - (1 + A_{11})]\, C_{n-3} + (n+1)^2 c_{n+1} + (n A_{12} + A_{24})\, c_n$$
$$+ \,[-2\,(n-2)(n-1) + (n-1)(A_{11}-1) + A_{23}] c_{n-1}$$
$$+ \,[A_{22} - (n-2)\, A_{12}] c_{n-2} + [(n-3)(n-4) - (n-3)\, A_{11} + A_{21}] c_{n-3} = 0,$$

or, as this may obviously be written

$$\Phi\,(n,\, c) + \frac{d\Phi\,(n,\, C)}{dn} = 0.$$

A simple application of the formulæ is to the differential equation

$$\frac{d^2 y}{dx^2} + \frac{1 - (\alpha + \beta + 1)\, x}{x\,(1-x)}\, \frac{dy}{dx} - \frac{\alpha\beta}{x\,(1-x)}\, y = 0,$$

of which a solution is the hypergeometric series $u = F(\alpha,\, \beta,\, 1,\, x)$. Here we have

$$A_{11} = -\,(\alpha + \beta + 1),\quad A_{12} = -\,(\alpha + \beta),$$
$$A_{21} = \alpha\beta,\quad A_{22} = \alpha\beta,\quad A_{23} = -\,\alpha\beta,\quad A_{24} = -\,\alpha\beta.$$

Applying the formula for $C_1,\, C_2,\, C_3 \ldots$ and writing $C_0 = 1$, we find readily

$$C_1 = \frac{\alpha \cdot \beta}{1.1},\quad C_2 = \frac{\alpha\,(\alpha+1)\,\beta\,(\beta+1)}{1.2.1.2},$$
$$C_3 = \frac{\alpha\,(\alpha+1)(\alpha+2)\,\beta\,(\beta+1)(\beta+2)}{1.2.3.1.2.3},\ \text{etc.}$$

Also for the coefficients $c_1,\, c_2 \ldots$

$$c_1 = \alpha\beta \left[c_0 + \frac{1}{\alpha} + \frac{1}{\beta} - 2\,(1) \right],$$
$$c_2 = \frac{\alpha\,(\alpha+1)\,\beta\,(\beta+1)}{1.2.1.2} \left[c_0 + \frac{1}{\alpha} + \frac{1}{\alpha+1} + \frac{1}{\beta} + \frac{1}{\beta+1} - 2\left(1 + \frac{1}{2}\right) \right],\ \text{etc.}$$

We have now found a system of fundamental integrals of the equation

$$\frac{d^2 y}{dx^2} + \frac{A_{11}x^2 + A_{12}x + A_{13}}{x\,(1-x^2)}\, \frac{dy}{dx} \, \frac{(A_{21}x^4 + A_{22}x^3 + A_{23}x^2 + A_{24}x + A_{25})}{x^2\,(1-x^2)^2}\, y = 0$$

in the domain of the point $x = 0$, and for this case the roots of the fundamental determinant equation were taken to be equal and each equal to zero, *i. e.* we made $A_{13} = 1$ and $A_{25} = 0$. Removing this restriction now, we have for the roots of the fundamental determinant equation

$$r_1 = \tfrac{1}{2}\{(1 - A_{13}) + \sqrt{(1 - A_{13})^2 - 4 A_{25}}\}$$
$$r_2 = \tfrac{1}{2}\{(1 - A_{13}) - \sqrt{(1 - A_{13})^2 - 4 A_{25}}\}$$

or say

$$r_1 = \alpha + \sqrt{\beta}$$
$$r_2 = \alpha - \sqrt{\beta}.$$

We will assume first that β, *i. e.* $(1 - A_{13})^2 - 4A_{25}$ is not equal to zero; now two cases arise, either $r_1 - r_2$ is zero or an integer or the same difference is neither zero nor an integer. By hypothesis the difference cannot be zero, so we have only to consider the cases when $\sqrt{(1 - A_{13})^2 - 4A_{25}}$ is and is not an integer.

Assume first that $\sqrt{(1 - A_{13})^2 - 4A_{25}}$ is not an integer, then the fundamental integrals will be of the form

$$y_1 = x^{r_1} \phi_1(x),$$
$$y_2 = x^{r_2} \phi_2(x).$$

Where $\phi_1(x)$ and $\phi_2(x)$ are uniform and continuous functions of x in the domain C_0 and are not equal to zero for $x = 0$. We have therefore

$$\phi_1(x), = u, = \sum_{n=0}^{n=\infty} \Delta_n x^n, \ \Delta_0 \ \text{not} = \text{zero}.$$

The differential equation for u is

$$\frac{d^2 u}{dx^2} = \frac{Q_1}{x} \frac{du}{dx} + \frac{Q_2}{x} u,$$

writing

$$Q_1'(x) = \frac{Q_1(x) - Q(0)}{x},$$

and substituting for Q_1 and Q_2 their values we have for the differential equation in u

$$x \frac{d^2 u}{dx^2} + (A_{13} + 2r) \frac{du}{dx} = - \left[\frac{x^2(A_{11} + A_{13} - 2r) + A_{12}x + 2r}{1 - x^2} \right] \frac{du}{dx}$$

$$- \left[\begin{array}{l} x^3 [A_{21} - rA_{11} + r(r-1)] + x^2 [A_{22} - rA_{12}] \\ \quad + x [A_{23} - rA_{13} + rA_{11} - 2r(r-1)] + [A_{24} + rA_{12}] \end{array} \right] \frac{u}{(1 - x^2)^2}.$$

Substituting now for u its value, and equating the coefficients of x^n, we have, after some easy reductions,

$$\begin{aligned}(n+1)(n+A_{13}) \Delta_{n+1} = & [-A_{21} + rA_{11} - r(r-1)] B_{n-3} + [rA_{12} - A_{22}] B_{n-2} \\ & + [rA_{13} - rA_{11} - A_{23} + 2r(r-1)] B_{n-1} - [A_{24} + rA_{12}] B_n \\ & + [-A_{11} - A_{13} + 2r] A_{n-2} - A_{12} A_{n-1} - 2r A_n,\end{aligned}$$

where

$$B_n = \sum_{k=0}^{k=n} \frac{1 + e^{(n+k)i\pi}}{2} \left(\frac{n-k}{2} + 1 \right) \Delta_k$$

$$A_n = \sum_{k=0}^{k=n} \frac{1 + e^{(n+k)i\pi}}{2} (k+1) \Delta_{k+1}.$$

(It will be noticed that the coefficient A_{25} does not appear explicitly in the above formula.)

In particular make $r = 0$, $A_{13} = 1$; then for $n = 0$ we have

$$\Delta_1 = - A_{24}\Delta_0,$$

for $n = 1$ $\qquad\qquad \Delta_2 = (- A_{23} + A_{24}^2 + A_{12}A_{24})\,\Delta_0,$ etc.,

results agreeing with those obtained above.

If we compute the constants Δ from the formula, they will all be found to be of the form $\qquad\qquad \Delta_n = \Gamma_n\Delta_0,$

then giving to r the values r_1 and r_2 we find the two functions above designated as $\phi_1(x)$ and $\phi_2(x)$: the fundamental integrals of the differential equation in the domain C_0 are therefore $\qquad y_1 = x^{r_1}\phi_1(x),$

$$y_2 = x^{r_2}\phi_2(x),$$

in the case where r_1 and r_2 are different and their difference is not an integer. A particular case of the above formula is when

$$A_{11} = -(\alpha + \beta + 1), \quad A_{12} = \gamma - \alpha - \beta - 1, \quad A_{13} = \gamma,$$
$$A_{21} = \alpha\beta, \quad A_{22} = \alpha\beta, \quad A_{23} = -\alpha\beta, \quad A_{24} = -\alpha\beta, \quad A_{25} = 0.$$

The roots of the fundamental determinant equation are now

$$r_1 = 0,$$
$$r_2 = 1 - \gamma,$$

and the differential equation is

$$\frac{d^2y}{dx^2} + \frac{\gamma - (\alpha + \beta + 1)x}{x(1-x)}\frac{dy}{dx} + \frac{\alpha\beta}{x(1-x)}\,y = 0,$$

which, when γ is not a negative integer, has for the integral corresponding to $r_1 = 0$ the hypergeometric series $F(\alpha, \beta, \gamma, x)$

and, corresponding to $r_2 = 1 - \gamma$, when $2 - \gamma$ is not a negative integer, the hypergeometric series

$$x^{1-\gamma}F(\alpha + 1 - \gamma, \ \beta + 1 - \gamma, \ 2 - \gamma, \ x).$$

Substituting the above values of the A's in the formula we have first for $r_1 = 0$ $\qquad\qquad \Delta_1 = \dfrac{\alpha\beta}{1.\gamma},$

$$\Delta_2 = \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{1.2.\gamma.\gamma + 1}, \text{ etc.}$$

For the case of $r_2 = 1 - \gamma$ the formula is similarly verified.

The relation between five consecutive coefficients is found by clearing the equation of fractions; doing this and substituting for n its value, we find

$\{n\,(n+1)+(n+1)(A_{13}+2r)+[A_{25}+rA_{13}+r\,(r-1)]\}\Delta_{n+1}$

$+\{nA_{12}+A_{24}+rA_{12}\}\,\Delta_n$

$+\{-2\,(n-1)(n-2)+(n-1)(A_{11}-A_{13}-4rA_{13})$

$+[A_{23}+r\,(A_{11}-A_{13})-2r\,(r-1)]\,\Delta_{n-1}$

$+\{-(n-2)\,A_{12}+A_{22}-rA_{12}\}\,\Delta_{n-2}$

$+\{(n-3)(n-4)+[-A_{11}+2r](n-3)+[A_{21}-rA_{11}+r\,(r-1)]\}\Delta_{n-3}=0\,.$

Take now the case where r_1-r_2 is an integer, *i. e.*

$$\sqrt{(1-A_{13})^2-4A_{25}}=\text{integer},$$

then of course $4A_{25}$ must be equal to the sum of two squares, one of which is an integer.

The two integrals now are of the form

$$y_1=x^{r_1}\phi_{11}$$
$$y_2=x^{r_2}\,[\phi_{21}+\phi_{22}\log x]\,,$$

or say

$$y_1=x^{r_1}u$$
$$y_2=x^{r_2}\,[v+u\log x]\,,$$

since ϕ_{22} only differs from ϕ_{11} by a constant factor, and since Cy_2 is an integral as well as y_2.

The first of these integrals is of course the same as the one obtained above. To find the second substitute in the differential equation: the coefficient of $\log x$ will be zero, and we have left

$$x\left\{\frac{d^2v}{dx^2}+\left[\frac{A_{11}x^2+A_{12}x+A_{13}}{x\,(1-x^2)}+\frac{2r}{x}\right]\frac{dv}{dx}\right.$$

$$+\left.\left[\frac{A_{21}x^4+A_{22}x^3+A_{23}x^2+A_{24}x+A_{25}}{x^2\,(1-x^2)^2}+\frac{r\,[A_{11}x^2+A_{12}x+A_{13}]}{x^2\,(1-x^2)}+\frac{r\,(r-1)}{x^2}\right]u\right\}$$

$$=-\left\{2\frac{du}{dx}+\left[\frac{2r-1}{x}+\frac{A_{11}x^2+A_{12}x+A_{13}}{x\,(1-x^2)}\right]u\right\}$$

or

$$x^2\frac{d^2v}{dx^2}+\frac{(A_{11}-2r)\,x^3+A_{12}x^2+(A_{13}+2r)\,x}{1-x^2}\frac{dv}{dx}$$

$$+\{x^4\,[A_{21}-rA_{11}+r\,(r-1)]+x^3\,[A_{22}-rA_{12}]+x^2\,[A_{23}+rA_{11}-rA_{13}-2r\,(r-1)]$$

$$+x\,[A_{24}+rA_{12}]\}\,\frac{1}{(1-x^2)^2}$$

$$=-2x\frac{du}{dx}+\frac{(A_{11}-2r+1)\,x^2+A_{12}x+A_{13}+2r-1}{1-x^2}\,u\,.$$

Multiplying out by $(1-x^2)^2$: writing

$$u=\sum_{n=0}^{n'=\infty}\Delta_nx^n,\quad v=\sum_{n=0}^{n=\infty}\delta_nx^n,$$

and equating coefficients of x^{n+1}, we have for the determination of the coefficients in v, the relation

$$\{n(n+1) + (n+1)(A_{13} + 2r) + [A_{25} + rA_{13} + r(r-1)]\}\,\delta_{n+1}$$
$$+\ \{nA_{12} + A_{24} + rA_{12}\}\,\delta_n$$
$$+\ \{-2(n-1)(n-2) + (n-1)(A_{11} - A_{13} - 4rA_{13})$$
$$+ A_{23} + r(A_{11} - A_{13}) - 2r(r-1)\}\,\delta_{n-1}$$
$$+\ \{-A_{12}(n-2) + A_{22} - rA_{12}\}\,\delta_{n-2}$$
$$+\ \{(n-3)(n-4) + [-A_{11} + 2r](n-3) + A_{21} - rA_{11} + r(r-1)\}\,\delta_{n-3}$$
$$=\ [-2(n+1) - (A_{13} + 2r - 1)]\,\Delta_{n+1} - A_{12}\Delta_n$$
$$+\ [4(n-1) - A_{11} + A_{13} + 4r - 2]\,\Delta_{n-1} + A_{12}\Delta_{n-2} - 2(n-3)\,\Delta_{n-3}\,.$$

Here we must of course replace r by r_2, in order to find the values of δ.

The relation satisfied by the five consecutive coefficients Δ_{n+1}, Δ_n, Δ_{n-1}, Δ_{n-2}, Δ_{n-3} is what the above equation becomes when on the left-hand side δ is replaced by Δ, and the right-hand side is made equal to zero; using the same notation as that employed when $r_1 = r_2 = 0$, this may be written

$$\Phi(n, r_1, \Delta) = 0,$$

and consequently the relation giving the δ's may be written

$$\Phi(n, r_2, \delta) + \frac{d}{dn}\Phi(n, r_2, \Delta) = 0.$$

(It will be observed that the coefficients of Δ_{n+1} and δ_{n+1} are integers, since $2r = 1 - A_{13} =$ integer.)

Consider now the domain C_1 of the point $x = 1$. The fundamental determinant equation for this point is

$$r(r-1) + r\left[\frac{A_{11} + A_{12} + A_{13}}{2}\right] + \frac{A_{21} + A_{22} + A_{23} + A_{24} + A_{25}}{4} = 0;$$

or writing

$$\frac{A_{11} + A_{12} + A_{13}}{2} = J_1,$$

$$\frac{A_{21} + A_{22} + A_{23} + A_{24} + A_{25}}{4} = J_2,$$

$$r^2 - r(1 - J_1) + J_2 = 0,$$

the roots of which are

$$r_1 = \tfrac{1}{2}\{1 - J_1 + \sqrt{(1 - J_1)^2 - 4J_2}\}$$
$$r_2 = \tfrac{1}{2}\{1 - J_1 - \sqrt{(1 - J_1)^2 - 4J_2}\}.$$

Suppose, first, $J_1 = 0$ and $J_2 = 0$, *i. e.*,

$$r_1 = 1,\ r_2 = 0.$$

The integrals of the differential equation are now of the form

$$y_1 = (x - 1)u$$
$$y_2 = v + u\log(x - 1),$$

where u and v are in the domain C_1 uniform and continuous functions of x, which for $x = 1$ do not vanish. We have then

$$u = \sum_{n=0}^{n=\infty} C_n (x-1)^n,$$

$$v = \sum_{n=0}^{n=\infty} c_n (x-1)^n.$$

The differential equation becomes now

$$\frac{d^2y}{dx^2} - \frac{A_{11}x + A_{11} + A_{12}}{x(1+x)}\frac{dy}{dx} - \left[A_{21}x^3 + (A_{21}+A_{22})x^2 + (A_{21}+A_{22}+A_{23})x \right.$$
$$\left. + \frac{(A_{21}+A_{22}+A_{23}+A_{24})}{x^2(x-1)(1+x)^2} \right] y,$$

or

$$\frac{d^2y}{dx^2} = \frac{A_{11}x - A_{13}}{x(x+1)}\frac{dy}{dx} + \frac{A_{21}x^3 + (A_{21}+A_{22})x^2 + (A_{21}+A_{22}+A_{23})x - A_{25}}{x^2(x-1)(x+1)^2} y.$$

Change the variable by the transformation

$$x = x' + 1.$$

Change also, for brevity, the notation for the coefficients, by writing

$$A_{11} = \alpha_1, \quad A_{11} - A_{13} = \delta_1,$$
$$A_{21} = \alpha_2, \; 4A_{21}+A_{22} = \beta_2, \; 6A_{21}+3A_{22}+A_{23} = \gamma_2, \; 3A_{21}+2A_{22}+A_{23}-A_{25} = \delta_2,$$

and the equation becomes

$$\frac{d^2y}{dx'^2} = \frac{\alpha_1 x' + \delta_1}{(x'+1)(x'+2)}\frac{dy}{dx'} + \frac{\alpha_2 x'^3 + \beta_2 x'^2 + \gamma_2 x' + \delta_2}{x'(x'+1)^2(x'+2)^2} y,$$

also

$$y_1 = x'u,$$

$$y_2 = v + u \log x',$$

$$u = \sum_{n=0}^{n=\infty} C_n x'^n,$$

$$v = \sum_{n=0}^{n=\infty} c_n x'^n.$$

For convenience, again write the equation in the form

$$\frac{d^2y}{dx^2} = \frac{P_1}{x'}\frac{dy}{dx'} + \frac{P_2}{x'^2} y,$$

where

$$P_1 = x' \frac{(\alpha_1 x' + \delta_1)}{(x'+1)(x'+2)},$$

$$P_2 = x' \frac{(\alpha_2 x'^3 + \beta_2 x'^2 + \gamma_2 x' + \delta_2)}{(x'+1)^2(x'+2)}.$$

It is at once seen that the roots of the fundamental determinant equation for the above differential equation are, in the domain of $x' = 0$, $r_1 = 1$, $r_2 = 0$, as they should be. Write now $y = x'u$; substituting in

$$\frac{d^2y}{dx'^2} = \frac{\alpha_1 x' + \delta_1}{(x'+1)(x'+2)} \frac{dy}{dx'} + \frac{\alpha_2 x'^3 + \beta_2 x'^2 + \gamma_2 x' + \delta_2}{x'(x'+1)^2(x'+2)^2} y,$$

we have for u the equation

$$\frac{d^2u}{dx^2} = \left[\frac{(\alpha_1 - 2)x^2 + (\delta_1 - 6)x - 4}{x(x+1)(x+2)} \right] \frac{du}{dx}$$
$$+ \left[\frac{(\alpha_1 + \alpha_2)x^4 + (\beta_1 + 3\alpha_1 + \delta_1)x^3 + (\gamma_1 + 2\alpha_1 + \delta_1)x^2 + (2\delta_1 + \delta_2)x}{x^2(x+1)^2(x+2)^2} \right] u.$$

I write, for convenience (as no confusion is likely to arise), x instead of x'.

Clearing this of fractions, substituting for u its value, and equating the coefficients of x^{n+1}, we find, without much difficulty, the relation

$$4(n+1)(n+2)C_{n+1} + \left[12n(n-1) - n(2\delta_1 - 24) - (2\delta_1 + \delta_2) \right] C_n$$
$$+ \left[13(n-1)(n-2) - (n-1)(2\alpha_1 + 3\delta_1 - 26) - (\gamma_2 + 2\alpha_1 + 3\delta_1) \right] C_{n-1}$$
$$+ \left[6(n-2)(n-3) - (n-2)(\delta_1 + 3\alpha_1 - 12) - (\beta_2 + 3\alpha_1 + \delta_1) \right] C_{n-2}$$
$$+ \left[(n-3)(n-4) - (\alpha_1 - 2)(n-3) - (\alpha_1 + \alpha_2) \right] C_{n-3} = 0.$$

As an example of this, take the case of

$$\alpha_1 = -1, \quad \delta_1 = -2,$$
$$\alpha_2 = -1, \quad \beta_2 = -4, \quad \gamma_2 = -4, \quad \delta_2 = 0.$$

The differential equation in y now becomes

$$\frac{d^2y}{dx^2} + \frac{1}{x+1} \frac{dy}{dx} + \frac{1}{(x+1)^2} y = 0,$$

the integrals of which are

$$y_1 = \sin \log(x+1),$$
$$y_2 = \cos \log(x+1).$$

The first of these is the one to be considered. Expanding $\sin \log(x+1)$ in a series going according to ascending powers of x, we have

$$\sin \log(x+1) = u_0 + u_1 x + u_2 x^2 + \ldots$$

where $\quad n(n+1)u_{n+1} + n(2n-1)u_n + (n^2 - 2n + 2)u_{n-1} = 0;$

further, since for $x = 0$ we have $\sin \log(x+1) = 0$, it follows that $u_0 = 0$.

The next few coefficients are u_1 (which is unity—but may obviously be replaced by an arbitrary constant in the expression for the integral), $u_2 = -\frac{1}{2}u_1$, $u_3 = \frac{1}{6}u_1$, $u_4 = 0$, $u_5 = -\frac{1}{12}u_1$, $u_6 = \frac{1}{8}u_1$, etc.

We ought now to have

$$C_0 = u_1, \quad C_1 = u_2, \quad C_2 = u_3, \quad C_3 = 0, \quad C_4 = u_5, \quad \text{etc.,}$$

and by substitution of the above values of $\alpha_1, \delta_1, \alpha_2, \ldots \delta_2$ in the formula, it is easily seen that these conditions are all satisfied. Another relation connecting the coefficients C in this particular case is found by expanding $\dfrac{1}{1+x}$ and $\dfrac{1}{(1+x)^2}$ in the differential equation

$$\frac{d^2(xu)}{dx^2} + \frac{1}{1+x}\frac{d(xu)}{dx} + \frac{xu}{(1+x)^2} = 0,$$

and equating to zero the coefficient of x^n: this is

$$(n+1)(n+2)C_{n+1} + \sum_{k=0}^{k=n-1}(-)^{n+k-1}\left[5(k+1)C_{k+1} + 2(n-k)C_k\right]$$
$$+ \sum_{k=0}^{k=n}(-)^{n+k}(n-k+1)C_k = 0,$$

or, replacing n by $n-1$ and C_n by u_{n+1}, we have for the relation connecting the coefficients $u_1, u_2, \ldots$

$$n(n+1)u_{n+1} + \sum_{k=0}^{k=n-2}(-)^{(n+k-2)}\left[5(k+1)u_{k+2} + 2(u-k+1)u_{k+1}\right]$$
$$+ \sum_{k=0}^{k=n-1}(-)^{n+k-1}(n-k)u_{k+1} = 0.$$

Having found the values of the coefficients C, and consequently the value of the function u, it is only necessary to replace x by $x-1$ in order to get the first integral $\qquad y_1 = (x-1)u$ of the original differential equation. The second integral is in general of the form $\qquad y_2 = v + u\log(x-1),$ or, changing the variable again, simply

$$y_2 = v + u\log x,$$

where u and v go according to ascending powers of x. In the simple case just noted, viz. $\qquad \dfrac{d^2y}{dx^2} + \dfrac{1}{(x+1)}\dfrac{dy}{dx} + \dfrac{1}{(x+1)^2}y = 0,$

there is evidently no logarithm—the two integrals being in fact, as already mentioned, $\qquad y_1 = \sin\log(x+1), \; (= xu),$
$$y_2 = \cos\log(x+1),$$
the coefficients in the developments being in each case connected by the relation
$$n(n+1)u_{n+1} + n(2n-1)u_n + (n^2 - 2n + 2)u_{n-1} = 0,$$
with, in the case of y_1, $u_0 = 0$.

To determine whether or not logarithms exist in the case at present treated, *i. e.* when the roots of the fundamental determinant equation are $r_1 = 1$, $r_2 = 0$, and where consequently $r_1 - r_2 = 1$, we form the differential equation satisfied by the second derivatives of the integrals y_1 and y_2; according as there exist or do not exist negative roots of the fundamental determinant equation belonging to this new differential equation, there exist or do not exist logarithms in the original equation in y.

Resuming now the differential equation

$$\frac{d^2y}{dx^2} = \frac{\alpha_1 x + \delta_1}{(x+1)(x+2)} \frac{dy}{dx} + \frac{\alpha_2 x^3 + \beta_2 x^2 + \gamma_2 x + \delta_2}{x(x+1)^3(x+2)^2} y,$$

write
$$y = v + u \log x$$

(here, as before, x is written for x'). After some easy reductions, we find for v the equation

$$[x(x+1)(x+2)]^2 \frac{d^2v}{dx^2} - x^2(x+1)(x+2)(\alpha_1 x + \delta_1)\frac{dv}{dx} - x(\alpha_2 x^3 + \beta_2 x^2 + \gamma_2 x + \delta_2)v$$

$$= -2x(x+1)^2(x+2)^2\frac{du}{dx} + [(x+1)^2(x+2)^2 + x(x+1)(x+2)(\alpha_1 x + \delta_1)]u.$$

Substituting for u and v their values, viz.,

$$u = \sum_{n=0}^{n=\infty} C_n x^n, \quad v = \sum_{n=0}^{n=\infty} c_n x^n,$$

and equating coefficients of x^{n+1}, we have for the relation connecting the coefficients C and c

$$4n(n+1)c_{n+1} + [12n(n-1) - 2\delta_1 n - \delta_2]c_n$$
$$+ [13(n-1)(n-2) - (n-1)(2\alpha_1 + 3\delta_1) - \gamma_2]c_{n-1}$$
$$+ [6(n-2)(n-3) - (n-2)(3\alpha_1 + \delta_1) - \beta_2]c_{n-2}$$
$$+ [(n-3)(n-4) - \alpha_1(n-3) - \alpha_2]c_{n-3}$$

$$= -4(2n+1)C_{n+1} + [-24n+12+2\delta_1]C_n + [-26(n-1)+13+2\alpha_1+3\delta_1]C_{n-1}$$
$$+ [-12(n-2)+6+3\alpha_1+\delta_1]C_{n-2} + [-2(n-3)+(1+\alpha_1)]C_{n-3}.$$

In the particular case already referred to, viz.

$$\frac{d^2y}{dx^2} + \frac{1}{(x+1)}\frac{dy}{dx} + \frac{1}{(x+1)^3}y = 0$$

(*i. e.* $\alpha_1 = -1$, $\delta_1 = -2$, $\alpha_2 = -1$, $\beta_2 = -4$, $\gamma_2 = -4$, $\delta_2 = 0$), the integral which we are in search of is simply the function v, consequently the values of the constants c are found by writing the left-hand member of the above equation $= 0$; it is easy to see then that the coefficients c of the function v are the same as those in the development of $\cos \log (x + 1)$.